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THE MEASUREMENT OF NON-LINEAR FORCES AND MOMENTS BY MEANS OF FREE FLIGHT TESTS

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ABSTRACT

It has been observed, that the behavior of missiles either moving under the influence of non-linear forces and moments or flying at large angles of yaw is frequently well described by curves of the same form as those generated by linear force systems and small angles of yaw. With this in mind an "equivalent" linear solution to the actual equations of yawing motion is obtained.

This equivalent linear solution has been used in the analysis of a wide variety of programs fired on BRL's Spark Ranges and considerable success has been experienced. Excellent internal consistency has been observed in measuring non-linear normal and Magnus forces and their moments and, in all cases where wind tunnel results were available, they were in good agreement with range results.

The application of this technique to the equally important problem of predicting yawing motion is described.

TABLE OF SYMBOLS*

A		axial moment of inertia
a	=	$K_{10}^2 + K_{20}^2$
a_i		coefficients in the drag equation (Equation (22))
B		transverse moment of inertia
b	=	$K_{10} K_{20}$
d		diameter
$\vec{e}_1, \vec{e}_2, \vec{e}_3$		unit vectors along axes of fixed-plane co-ordinates
$\vec{F}$	=	(F_1, F_2, F_3) aerodynamic force vector
F_D		drag force
F_L		lift force
g		acceleration due to gravity
H	=	$\ell J_L - J_D + k_2^{-2} (J_H - \ell J_{MA})$
$I_1(p)$	=	$\int_0^p \int_0^q \delta^2 \, dr \, dq$
J_i	=	$\frac{\rho d^3}{m} K_i$
K_i	=	$\sum_{k=0} K_{i_{\delta^{2k}}} \delta^{2k}$; i = A, D, DA, F, H, L, M, MA, N, NA, S, T (Expansion as a function of δ^2 of aerodynamic coefficients defined in Equations (5 - 10))

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* Only those symbols which appear in the body of this report are listed here. Symbols which are introduced in the appendices appear close to their definitions.

$K_i \quad = \quad K_{i0}\, e^{-\alpha_i p}$; i = 1, 2 amplitude of i-th frequency

K_{i0} i = 1,2 mid-range amplitude of i-th frequency (p = 0).

K_A axial spin deceleration coefficients

K_D drag coefficient

K_{DA} axial drag coefficient

K_F Magnus force coefficient

K_H moment coefficient due to cross angular velocity

K_L lift force coefficient due to yaw

K_M overturning (or righting) moment coefficient due to yaw

K_{MA} moment coefficient due to cross acceleration

K_N normal force coefficient due to yaw

K_{NA} normal force coefficient due to cross acceleration

K_S normal force coefficient due to cross angular velocity

K_T Magnus moment coefficient

$k_1 \quad = \quad \sqrt{\frac{A}{md^2}}$ axial radius of gyration in calibers

$k_2 \quad = \quad \sqrt{\frac{B}{md^2}}$ transverse radius of gyration in calibers

$\ell \quad = \quad \frac{u_1}{u}$ cosine of total yaw angle

$$M \quad = \quad \ell k_2^{-2} J_M = \sum_{k=0}^{\infty} \ell M_{2k}\, \delta^{2k} = \sum_{k=0}^{\infty} M^*_{2k}\, \delta^{2k}$$

$\vec{M}$	=	(M_1, M_2, M_3) aerodynamic moment vector
m		mass
(n_1, n_2, n_3)		direction cosines of missile's axis with respect to range co-ordinates
p	=	$\int_o^t \frac{u}{d}\,dt$ arclength along the trajectory in calibers
Q_i		constants in the swerve equations (Equations (82 -83))
q		change in center of mass location measured in calibers (Eqs. (99 -100))
T	=	$\ell(J_L - k_1^{-2} J_T) = \sum_{k=0}^{\infty} T^*_{2k}\, \delta^{2k}$
t		time
$\vec{u}$	=	(u_1, u_2, u_3) velocity vector
u	=	$\lvert\vec{u}\rvert$ magnitude of the velocity
u_o		magnitude of the velocity at mid-range
(x_1, x_2, x_3)		vector in the range co-ordinate system
(y_1, y_2, y_3)		vector in the fixed-plane co-ordinate system
α_i		exponential damping coefficient of i-frequency
δ	=	$\sqrt{\lambda\bar{\lambda}}$ magnitude of the sine of the total yaw angle

$$\left.\begin{aligned}
\delta_e^2 &= K_{10}^2 + K_{20}^2 + \frac{K_{10}^2\phi_1' - K_{20}^2\phi_2'}{\phi_1' - \phi_2'} \\
\delta_{e1}^2 &= K_{10}^2 + 2K_{20}^2 \\
\delta_{e2}^2 &= K_{20}^2 + 2K_{10}^2
\end{aligned}\right\} \text{Effective squared yaws}$$

δ_m		maximum value of δ
θ		orientation angle of plane of yaw
λ	=	$\lambda_2 + i\lambda_3 = \dfrac{u_2 + iu_3}{u}$ complex yaw
μ	=	$\mu_2 + i\mu_3 = \dfrac{(\omega_2 + i\omega_3)d}{u}$ dimensionless complex cross angular velocity
ν	=	$\dfrac{\omega_1 d}{u}$ dimensionless missile spin
$\bar{\nu}$	=	$\dfrac{A}{B}\nu$
$\hat{\nu}$	=	$\dfrac{\Omega_1 d}{u}$ dimensionless co-ordinate system spin
$[\hat{\nu}]$		average value of $\hat{\nu}$
ρ		air density
ϕ_i	=	$\phi_{i0} + \phi_i{}' p$ phase angle of i-frequency
ϕ'	=	$\phi_1{}' = -\phi_2{}'$
ϕ_o	=	$\phi_{10} - \theta = \theta - \phi_{20}$
ψ	=	$\arccos n_3$
$\vec{\Omega}$	=	$(\Omega_1, \Omega_2, \Omega_3)$ angular velocity vector of co-ordinate system
$\vec{\omega}$	=	$(\omega_1, \omega_2, \omega_3)$ angular velocity vector of missile
$\dot{(\ )}$	=	$\dfrac{d}{dt}(\ )$
$(\ )'$	=	$\dfrac{d}{dp}(\ )$
$\wedge$		circumflex superscript with the exception of $\hat{\nu}$ denotes quantities in the fixed-plane co-ordinate system
$\sim$		tilde superscript with the exception of δ^2 denotes quantities appearing in the solution of the <u>linearized</u> yaw equation
*		asterisk denotes the modification of quantities involving aerodynamic coefficients through the consideration of the cosine of the yaw angle, ℓ.

1. INTRODUCTION

The free flight spark range technique measures the aerodynamic forces and moments acting on a missile by means of very accurate observations of its motion in flight. This process requires a knowledge of the functional dependence of these forces and moments on the dynamic variables of the motion in order that the solution curves to the differential equations of motion may be obtained. These solution curves are fitted to the motion and the forces and moments calculated from the parameters of the fit.

This need for solutions in closed form has traditionally limited range tests to motions which are described by linear equations. Since non-linear terms arise from both the size of the motion (non-linear geometry terms) and the presence of second order or higher terms in the aerodynamic force expansion (non-linear force terms), this means that the range technique is restricted to configurations possessing linear force systems and flying at small angles of yaw.

Strangely enough, a number of models have been fired in the BRL Ranges which either possessed known non-linearities in their force systems or flew at large angles of yaw and it was found that their motion could be very well fitted by functions which were solutions of the linearized equations. This seemed to imply that the parameters of these linear equations should be "average values" of the coefficients of the parent non-linear equation. It is the purpose of this report first to derive the equations which relate these average values to the non-linear force terms and certain characteristics of the motion and then to demonstrate the great value of these relations by applying them to a number of programs which have been fired at BRL. The success of this technique more than doubles the values of free flight ranges for both the ballistician and the aerodynamicist.

Finally the extension of this method to the even more important problem of the prediction of yawing motion is described.

2. DYNAMIC CONSIDERATIONS

In order to investigate the yawing motion of missiles moving at large angles of yaw, the exact equations of motion must be derived. An important feature of this derivation lies in the proper selection of the co-ordinate system and the dynamic variables. This selection should be made so that the resulting equations are as simple as possible and reasonably compatible with the basic assumption that their solution may be approximated by a solution of the linearized equations. In other words the non-linear equation must possess the same general characteristics of the linearized equation, i.e., order of equations, number of variables, symmetry of variables, etc.

For arbitrary rigid bodies the best co-ordinate system is one whose axes lie along the body's principal axes of inertia. Since the mass distributions of most missiles are rotationally symmetrical about the longitudinal axis of inertia, all transverse moments of inertia are assumed to be equal and all axes perpendicular to the longitudinal axis are principal axes of inertia. A right-handed Cartesian co-ordinate system with numbered axes will be constrained so that its 1-axis is aligned with the missile's longitudinal axis and pointing forward.

$\vec{\Omega}$, the angular velocity vector of the co-ordinate system relative to an inertial system, will have components $(\Omega_1, \Omega_2, \Omega_3)$ while the missile's angular velocity vector, $\vec{\omega}$, will have components $(\omega_1, \omega_2, \omega_3)$. This definition of the co-ordinate system, then, requires that $\Omega_2 = \omega_2$ and $\Omega_3 = \omega_3$. If Ω_1 and an initial orientation of the 2-axis are specified, the co-ordinate system would be completely determined. In Appendix A the equations of motion are derived for arbitrary values of Ω_1 and we see that these equations would be greatly simplified if $\Omega_1 = 0$. For this reason we will make considerable use of a non-spinning co-ordinate system for which the 2-axis initially lies in the horizontal plane pointing to the left and whose axial spin, Ω_1, is identically zero.

Turning now to the question of dependent variables it may be seen that the major contenders are the Eulerian angles and the direction cosines. The question of compatability now appears. Although the linearized equations in terms of Eulerian angles are symmetric in the components of yaw, the corresponding exact equations cannot be. This is clear from the definition of the angles. For this reason* we will express our equations in terms of direction cosines.

At this point we must precisely define our variables. The yaw angle is defined to be the angle between the missile's axis and the tangent to its trajectory. If the missile's velocity vector, $\vec{u}$, has components (u_1, u_2, u_3) and magnitude, u, the sine of the yaw angle is $\frac{\sqrt{u_2^2 + u_3^2}}{u}$ and a unit vector lying in the plane of yaw has direction cosines $\left(0, \frac{u_2}{\sqrt{u_2^2 + u_3^2}}, \frac{u_3}{\sqrt{u_2^2 + u_3^2}}\right)$. Since we will consider only those missiles possessing trigonal or greater rotational symmetry, it will be convenient to represent quantities in the plane normal to the missile's axis by complex numbers. With this in mind we define the complex yaw vector, λ, to be a vector which lies in the plane of yaw and whose magnitude is the sine**of the yaw angle.

$$\therefore \quad \lambda = \frac{u_2 + iu_3}{u} \tag{1}$$

The cosine of the yaw angle will be needed in this report and will be denoted by ℓ.

$$\therefore \quad \ell = \frac{u_1}{u} \tag{2}$$

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* As can be seen from an examination of Reference 1 the Eulerian angles also introduce considerable algebraic complexity.

** In Reference 2, u_1 is used as a characteristic velocity for forming dimensionless quantities and, hence, the complex yaw of that reference is proportional to the tangent of the yaw angle. Although this use of u_1 does simplify the center of mass relations, it also introduces a number of complications which our choice of u avoids.

In a similar fashion the axial and transverse components of the angular velocity may be separated and written in complex form.

$$\nu = \frac{\omega_1 d}{u}$$

$$\mu = \frac{(\omega_2 + i\omega_3)d}{u} \tag{3}$$

where d is diameter.

Finally we will select our independent variable to be distance measured along the trajectory in calibers. If this variable is identified by p,

$$p = \int_0^t \frac{u}{d}\, dt \tag{4}$$

where t is time.

The generalized[3] linear expansion of the aerodynamic force and moment assumes that the force and moment are linear functions of λ, μ, and their derivatives in a non-rolling co-ordinate system. For symmetric missiles this assumption introduces eighteen coefficients. If we limit ourselves to only those coefficients having a measurable effect on the motion, this total reduces to ten and we have the following expansion.

$$F_1 = - \rho\, d^2\, u^2\, K_{DA} \tag{5}$$

$$F_2 + iF_3 = \rho\, d^2\, u^2 \left[(- K_N + i\nu K_F)\lambda + iK_S\mu - K_{NA}(\lambda' + i\hat{\nu}\lambda) \right] \tag{6}$$

$$M_1 = - \rho d^3 u^2\, \nu K_A \tag{7}$$

$$M_2 + iM_3 = \rho d^3 u^2 \left[(-\nu K_T - iK_M)\lambda - K_H\mu - iK_{MA}(\lambda' + i\hat{\nu}\lambda) \right] \tag{8}$$

where (F_1, F_2, F_3) are components of the aerodynamic force,

(M_1, M_2, M_3) are components of the aerodynamic moment,

ρ is air density,

$\hat{\nu} = \dfrac{\Omega_1 d}{u}$, and

K_i are dimensionless aerodynamic coefficients,

$\lambda' = \dfrac{d\lambda}{dp}$

For this report we will only consider non-linearities caused by the size of the yaw angle. Since symmetry[4] requires that the aerodynamic coefficients must be functions of $\lambda\bar{\lambda} = \delta^2$, we will only consider such a dependency on the square of the sine of the yaw angle.

Two other force coefficients will be used in this report. These are based on a resolution of the aerodynamic force in the plane of yaw into components along the trajectory and perpendicular to it. If F_D is the component along the trajectory and F_L the component perpendicular to the trajectory and pointing toward the missile's nose,

$$F_D = -\rho d^2 u^2 K_D \tag{9}$$

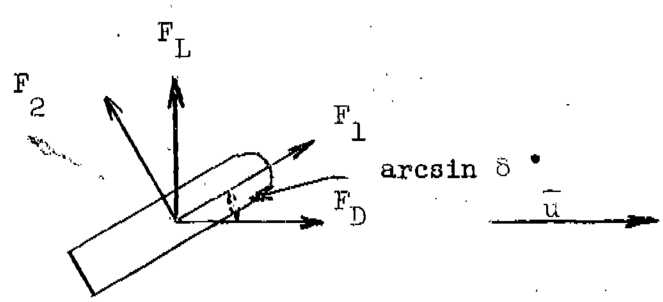
$$F_L = -\rho d^2 u^2 K_L \delta \tag{10}$$

where $\delta = \sqrt{\lambda\bar{\lambda}}$ (δ is the magnitude of the sine of the yaw angle.)

In order to obtain relations between these two new coefficients and those defined in Equations (5 - 6), we require that the 2-axis be in the plane of yaw so that λ is real and equal to $-\delta$.

$$\therefore \quad F_D = \ell F_1 - \delta F_2 \tag{11}$$

$$F_L = \delta F_1 + \ell F_2 \tag{12}$$



Substituting Equations (5), (6), (9), and (10) in Equations (11) (12) and neglecting the K_S and K_{NA} terms.

$$K_D = \ell K_{DA} + \delta^2 K_N \tag{13}$$

$$K_L = \ell K_N - K_{DA} \tag{14}$$

From Equations (13) (14) the usual small yaw approximations follow:

$$K_D \doteq K_{DA} \tag{15}$$

$$K_L \doteq K_N - K_{DA} \tag{16}$$

Finally if K_{DA} is eliminated from Equations (13-14) the following useful relation results:

$$\ell K_L = K_N - K_D \tag{17}$$

In Appendix A, the exact equations of motion for the aerodynamic force and moment as defined by Equations (5 - 8) are derived. (Eqs. (A3, A10, A19)). If the effect of gravity and the variation of spin are neglected ($J_g = \gamma = D = 0$) they may be written in the following form in the non-spinning co-ordinate system ($\hat{\nu} = 0$)

$$u' = - J_D u \tag{18}$$

$$\nu' = 0 \tag{19}$$

$$\lambda'' + \left[H - \frac{\ell'}{\ell} - i\bar{\nu}\right] \lambda' + \left[- M + \ell J_L' - i\bar{\nu}T\right] \lambda = 0 \tag{20}$$

where

$$H = \ell\, J_L - J_D + k_2^{-2}(J_H - \ell\, J_{MA})$$

$$\bar{\nu} = \frac{A}{B}\nu$$

$$M = \ell k_2^{-2} J_M$$

$$T = \ell(J_L - k_1^{-2} J_T)$$

A is axial moment of inertia

B is transverse moment of inertia

k_1 is axial radius of gyration in calibers

k_2 is transverse radius of gyration in calibers

$$J_i = \frac{\rho d^3}{m} K_i$$

m is mass.

3. SMALL YAWING MOTION WITH NON-LINEAR MOMENT AND NO DAMPING

In this section a very simple example of non-linear yawing motion will be considered in some detail. This is the case of a missile flying at small yaw and acted on by a cubic overturning or restoring moment. Later the effect of other non-linear forces and moments as well as the geometric non-linearities will be considered.

The basic feature of all the theoretical results of this report is the assumption that over "small" sections of a missile's trajectory the non-linearities present in the equations of motion do not cause the motion to be qualitatively different from motion based on the linearized equations. For example, the yawing motion of a symmetric missile acted on by non-linear forces and moments should still be epicyclic when a "small" portion of the trajectory is considered. The parameters of the epicycle, however, would probably be related to the size of the motion. This assumption seems to be reasonable when the non-linearities are themselves "small". The experience of ballisticians as based on actual free flight tests have indicated that relatively long sections of trajectories and large non-linearities are still "small" enough for this assumption.

In order to illustrate this point the data analysis of a non-linear drag force will be outlined. Since this is the only non-linear aerodynamic force which up to the present time has been successfully handled by ballisticians, this digression will also provide a good background for later derivations.

The non-linear dependence of the drag force on the magnitude of the yaw is very well described by the assumption that the drag coefficient is a quadratic function of δ where δ is the sine of the total yaw angle.

$$\therefore K_D = K_{D_0} + K_{D_{\delta^2}} \delta^2 \qquad (21)$$

In Reference 5 it is shown that the usual drag reduction for flat trajectories* involves the fitting of the time measurements to a cubic in distance.

$$t = a_0 + a_1 p + a_2 p^2 + a_3 p^3 \qquad (22)$$

where p is measured from the middle of the observed trajectory. It is further shown there that under certain reasonable approximations the following equation applies for drag force of the same form as Equation (21).

$$t = t_0 + \frac{d}{u_0} p + \frac{d}{u_0} J_{D0} \frac{p^2}{2} + a_3 p^3 + \frac{d}{u_0} J_{D_\delta 2} I_1 (p) \qquad (23)$$

where u_0 is velocity at midrange ($p = 0$),

a_3 is a constant related to drag dependence on Mach number,

$I_1 (p) = \int_0^p \int_0^q \delta^2 \, dr \, dq$, and

$J_{D_i} = \frac{\rho d^3}{m} K_{D_i}$.

If the yaw is well described by an epicycle, then

$$\lambda = K_1 e^{i\phi_1} + K_2 e^{i\phi_2} \qquad (24)$$

where $K_j = K_{j0} e^{-\alpha_j p}$; $\phi_j = \phi_{j0} + \phi_j{}' p$, and

K_{j0}, α_j, ϕ_{j0}, ϕ_{j0}' are real constants.

$$\therefore \quad \delta^2 = \lambda\bar{\lambda} = K_1^2 + K_2^2 + K_1 K_2 \left(e^{i(\phi_1 - \phi_2)} + e^{i(\phi_2 - \phi_1)}\right) \qquad (25)$$

Although the expression in parentheses may be given more simply by $2 \cos(\phi_1 - \phi_2)$, this form will be more convenient for the non-linear yaw equation. For the case of zero yaw drag coefficient, we see, from Equations (22) and (23), that

$$K_D = K_{D0} = \frac{m}{\rho d^3} \quad \frac{2a_2}{a_1} \qquad (26)$$

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* A trajectory is said to be flat when the component of the gravational force along the trajectory does not change. The yaw drag treatment may be extended to treat non-flat trajectories if this is necessary.

When the yaw-drag coefficient has a measurable effect, it does not usually affect the quality of this cubic fit for segments of trajectories as long as 10,000 calibers. It has been found that in this case the same configuration flying at different average yaws will provide good cubic fits but different values of $\frac{2a_2}{a_1}$. The problem is then to find the quadratic contribution of the yaw-drag term of Equation (23).

Since all drag reductions contain a number of periods of the cosine term, the cubic fit of the time history effectively neglects this component of δ^2. With this in mind we look for the "best" quadratic approximation to $I_1(p)$ over the length of observed trajectory, L. To do this we define an average squared yaw, $\widetilde{\delta^2}$, so that the integral

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \left[I_1(p) - \widetilde{\delta^2} \left(\frac{p^2}{2}\right) \right]^2 dp$$

is a minimum.* $\widetilde{\delta^2}$ is, therefore, the best quadratic coefficient from the standpoint of least squares. Differentiating the integral with respect to $\widetilde{\delta^2}$ and setting the result equal to zero.

$$\widetilde{\delta^2} = \frac{\int_{-\frac{L}{2}}^{\frac{L}{2}} p^2 I_1(p)\, dp}{\int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{1}{2}\right) p^4\, dp} .$$

* Note that this average squared yaw is different from the more traditionally used mean squared yaw $\overline{\delta^2} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} (K_1^2 + K_2^2)\, dp$. Although the average squared yaw has better theoretical justification, the numerical difference is usually not important.

$$= \frac{5}{(\frac{L}{2})^5} \int_{-\frac{L}{2}}^{\frac{L}{2}} p^2 \, I_1(p) \, dp \qquad (27)$$

where $I_1(p) = \int_0^p \int_0^q (K_1^{\,2} + K_2^{\,2}) \, dr \, dq$

$$= K_{10}^{\,2} \left[\frac{e^{-2\alpha_1 p} - 1 + 2\alpha_1 p}{(2\alpha_1)^2} \right] + K_{20}^{\,2} \left[\frac{e^{-2\alpha_2 p} - 1 + 2\alpha_2 p}{(2\alpha_2)^2} \right]$$

The complete expansion for $\tilde{\delta}^2$ may be computed from the power series expansion of $I_1(p)$. Since three non-zero terms are usually sufficient, the following expression for $\tilde{\delta}^2$ is easily derivable.

$$\tilde{\delta}^2 = 10 \left\{ K_{10}^{\,2} \left[\frac{1}{5(2!)} + \frac{(\alpha_1 L)^2}{7(4!)} + \frac{(\alpha_1 L)^4}{9(6!)} \right] + K_{20}^{\,2} \left[\frac{1}{5(2!)} + \frac{(\alpha_2 L)^2}{7(4!)} + \frac{(\alpha_2 L)^4}{9(6!)} \right] \right\} \qquad (27')$$

Returning to equation (26),

$$K_{D_{range}} = \frac{m}{\rho d^3} \frac{2a_2}{a_1} = K_{D0} + K_{D_{\delta^2}} \tilde{\delta}^2 \qquad (28)$$

In most cases a good approximation for $\tilde{\delta}^2$ can be obtained by taking the first term in the expansion of Eq. (27').

$$\therefore K_{D_{range}} = K_{D0} + K_{D_{\delta^2}} (K_{10}^{\,2} + K_{20}^{\,2}) \qquad (29)$$

The value of this relation is clearly shown in Figure 1 where $K_{D_{range}}$ is plotted versus $\tilde{\delta}^2$ for a body of revolution fired at yaw angles up to 30°. A further check on this technique lies in the good agreement of the wind tunnel value of $K_{D_{\delta^2}} = 1.85$ with the free flight value of 1.82. (See Table IV on Page 45.)

Free flight tests[6] have shown that the situation for the yawing motion is quite similar to this yaw drag case. Although the non-linearities do not measureably change the nature of the epicyclic motion which is predicted by the linear theory, they do affect the values of the epicyclic parameters. It is, therefore, necessary to obtain relations of the same type as Eq. (29).

Since the simple case of the small amplitude yawing motion of a missile acted on by a cubic static moment will first be considered, Equation (20) can be considerably simplified. The small amplitude assumption implies the approximations $\ell = 1$ and $\ell' = 0$ while the restriction to a static moment eliminates the terms in H, J_L' and T. The equation under consideration, therefore, becomes

$$\lambda'' - i\bar{\nu}\lambda' - (M_0 + M_2\,\delta^2)\lambda = 0 \tag{30}$$

where $M_0 = \frac{\rho d^5}{B} K_{M_0}$

$M_2 = \frac{\rho d^5}{B} K_{M_{\delta^2}}$

$K_M = K_{M0} + K_{M_{\delta^2}}\,\delta^2$

$\delta^2 = \lambda\bar{\lambda}$.

The solution to the linearized form of Equation (30) for a gyroscopically stable missile* is an epicycle without damping.

$$\lambda = K_{10}\, e^{i\tilde{\phi}_1} + K_{20}\, e^{i\tilde{\phi}_2} \tag{31}$$

* A missile is gyroscopically stable if $\bar{\nu}^2 > 4M_0$. All statically stable missiles ($M_0 < 0$), for example, are gyroscopically stable. Statically unstable missiles ($M_0 > 0$) are gyroscopically stable when their gyroscopic stability factors, $s = \frac{\bar{\nu}^2}{4M_0}$, are greater than unity.

where * $\tilde{\phi} = \tilde{\phi}_{i0} + \tilde{\phi}_i' p$

$\tilde{\phi}_1'$ $\tilde{\phi}_2'$, and

K_{i0}, $\tilde{\phi}_{i0}$, and $\tilde{\phi}_1'$ are real constants.

In Equation (31) the K_{i0} and $\tilde{\phi}_{i0}$ s depend on the initial conditions but the frequencies, $\tilde{\phi}_i'$ are functions fo the coefficients of the linearized equation and are independent of the initial conditions. The well known relations for the frequencies are

$$\tilde{\phi}_1' + \tilde{\phi}_2' = \bar{\nu} \tag{32}$$

$$\tilde{\phi}_1' \cdot \tilde{\phi}_2' = M_0 \tag{33}$$

It will be shown, however, that the frequencies of the quasi-linear solution of Equation (30) do depend on the initial conditions.

If the solution to Equation (30) is assumed to be of the same form as Equation (31), it can be substituted in Equation (30) to provide the following equation

$$\begin{aligned} & K_{10}\, e^{i\phi_1}\left[- \phi_1'^2 + \bar{\nu}\,\phi_1' - M_0 - M_2\,(K_{10}^{\ 2} + 2K_{20}^{\ 2}) \right] \\ & + K_{20}\, e^{i\phi_2}\left[- \phi_2'^2 + \bar{\nu}\,\phi_2' - M_0 - M_2\,(K_{20}^{\ 2} + 2K_{10}^{\ 2}) \right] \\ & - M_2\left[K_{10}^{\ 2}\, K_{20}\, e^{i(2\phi_1 - \phi_2)} + K_{10}\, K_{20}^{\ 2}\, e^{i(2\phi_2 - \phi_1)} \right] = 0. \end{aligned} \tag{34}$$

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* The tilde superscripts are used to emphasize that the $\tilde{\phi}_i'$ s appear in the solution to the <u>linearized</u> equation.

Since we know that the epicycle is a good description of the actual motion, the third term in Equation (34) which contains mixed frequencies is neglected and the following two equations may be obtained.

$$\phi_1' (\bar{\nu} - \phi_1') = M_0 + M_2 \delta_{e1}^2 \tag{35}$$

$$\phi_2' (\bar{\nu} - \phi_2') = M_0 + M_2 \delta_{e2}^2 \tag{36}$$

where $\delta_{e1}^2 = K_{10}^{\ 2} + 2K_{20}^{\ 2}$

$$\delta_{e2}^2 = K_{20}^{\ 2} + 2K_{10}^{\ 2}$$

Unfortunately Equations (35 - 36) are both quadratic equations and two different values of each frequency are possible. This difficulty, however, can be resolved. Since we are considering solutions close to the solution of the linearized equation, the solution of Equation (35) close to $\tilde{\phi}_1'$ should be selected and similarly for Equation (36). This means that the larger root of Equation (35) and the smaller root of Equation (36) should be used.

Equations (35 - 36) show that δ_{ei}^2, the effective value of the squared yaw for the i - th frequency, is twice as sensitive to the amplitude of other frequency as it is to its own amplitude. The effect comes from the cosine term which was omitted from the yaw drag analysis. For the yawing motion this periodic part of δ^2 can not be neglected.

In order to obtain relations similar to Equations (32 - 33), M_0 is first eliminated between Equations (35 - 36) and then $\bar{\nu}$ is eliminated between them

$$\therefore \quad \phi_1' + \phi_2' = \bar{\nu} + M_2 \left(\frac{K_{10}^{\ 2} - K_{20}^{\ 2}}{\phi_1' - \phi_2'} \right) \tag{37}$$

$$\phi_1' \cdot \phi_2' = M_0 + M_2 \delta_e^{\ 2} \tag{38a}$$

$$\delta_e^{\ 2} = \frac{\phi_1' \delta_{e2}^2 - \phi_2' \delta_{e1}^2}{\phi_1' - \phi_2'}$$

$$= K_{10}^2 + K_{20}^2 + \frac{\phi_1' K_{10}^2 - \phi_2' K_{20}^2}{\phi_1' - \phi_2'} \tag{38b}$$

At first glance this derivation of our "equivalent" solution seems to be surrounded by an atmosphere of expediency and to be resting on empirical assumptions. Actually this is not true. As we shall now show our technique has a good theoretical background and requires a minimum algebraic work in comparison with other methods. Finally it will be shown that the technique is not restricted to non-linearities proportional to δ^2 but can be easily extended to polynomial functions of δ^2.

The derivation of Equations (35 - 36) may actually be considered the first step of an iteration procedure. It is certainly reasonable to assume for small non-linearities that the first step of the iteration has the same form as the linear solution. The error term,

$$M_2 \left[K_{10}^2 K_{20} e^{i(2\phi_1 - \phi_2)} + K_{10} K_{20}^2 e^{i(2\phi_2 - \phi_1)} \right],$$

introduces mixed frequencies which are a characteristic of a non-linear equation. The next step in the iteration would be to assume a solution of the form

$$\lambda = K_{10} e^{i\phi_1} + K_{20} e^{i\phi_2} + K_{30} e^{i(2\phi_1 - \phi_2)} + K_{40} e^{i(2\phi_2 - \phi_1)} \tag{39}$$

Substitution of Equation (39) in Equation (30) would provide four complicated equations in terms of the four K_{i0}'s and two frequencies ϕ_i's, and our error term would then be of the form

$$M_2 (A_1 e^{i(3\phi_1 - 2\phi_2)} + A_2 e^{i(3\phi_2 - 2\phi_1)} + A_3 e^{i(4\phi_1 - 3\phi_2)} + A_4 e^{i(4\phi_2 - 3\phi_1)}).$$

where the A_i are fifth order combinations of the K_{i0}'s. This process may then be further iterated to yield a series expansion of the almost periodic solution of Equation (30). Fortunately as we shall see, the experimental results of this report show that only the first step of this process is needed.

Another method of treating Equation (30) which may seem to be more elegant than the method of this report is that of Kryloff and Bogoliuboff.* It will be shown that this method provides the same results as our direct substitution approach and requires more algebraic work. Kryloff and Bogoliuboff move the non-linear terms to the right-hand side of the equation and place the solution of the linearized equation in these terms so that they are functions of the independent variable p. The method of variation of parameters is then used to solve the resulting inhomogeneous linear equation. In order to solve the differential equations for the parametric functions, the terms arising from the inhomogeneous term of the original equation are averaged over the two periods of the motion.

As an illustration of this method we rewrite Equation (30) as

$$\lambda'' - i\bar{\nu}\lambda - M_o\lambda = M_2\ \delta^2\ \lambda = M_2\ f(p) \tag{40}$$

where $f(p) = (K_{10}{}^2 + 2K_{20}{}^2)K_{10}\ e^{i\tilde{\phi}_1} + (K_{20}{}^2 + 2K_{10}{}^2)K_{20}\ e^{i\tilde{\phi}_2}$

$$+ K_{10}{}^2K_{20}\ e^{i(2\tilde{\phi}_1 - \tilde{\phi}_2)} + K_{20}{}^2\ K_{10}\ e^{i(2\tilde{\phi}_2 - \tilde{\phi}_1)}$$

The parameters of this solution are the magnitudes K_{10} and the phase angles $\tilde{\phi}_{10}$'s. Differentiating Equation (31) we have

$$\lambda' = i\tilde{\phi}_1'\ K_{10}\ e^{i\tilde{\phi}_1} + i\tilde{\phi}_2'\ K_{20}\ e^{i\tilde{\phi}_2} + (K_{10}' + i\tilde{\phi}_{10}'\ K_{10})e^{i\tilde{\phi}_1} + (K_{20}' + i\tilde{\phi}_{20}'K_{20})e^{i\tilde{\phi}_2}$$

If the last two terms are set equal to zero,

$$(K_{10}' + i\tilde{\phi}_{10}'K_{10})e^{i\tilde{\phi}_1} + (K_{20}' + i\tilde{\phi}_{20}'K_{20})e^{i\tilde{\phi}_2} = 0, \tag{42}$$

* In Reference 7 it is shown that this method is fundamentally the same as that of Van der Pol. They differ in the fact that Kryloff and Bogoliuboff express the linear solution in polar co-ordinate form and Van der Pol expresses it in Cartesian form.

then Equation (41) may be differentiated again to yield

$$\lambda'' = -\tilde{\phi}_1'^{\,2} K_{10} e^{i\tilde{\phi}_1} - \tilde{\phi}_2'^{\,2} K_{20} e^{i\tilde{\phi}_2} + i\tilde{\phi}_1' (K_{10}' + i\tilde{\phi}_{10}' K_{10}) e^{i\tilde{\phi}_1}$$

$$+ i\tilde{\phi}_2' (K_{20}' + i\tilde{\phi}_{20}' K_{20}) e^{i\tilde{\phi}_2}. \tag{43}$$

Substituting Equations (31, 41, 43) for λ, λ', λ'' in Equation (40) and solving for $K_{10}' + i\tilde{\phi}_{10}' K_{10}$ by use of Equation (42),

$$K_{10}' + i\tilde{\phi}_{10}' K_{10} = \frac{M_2 f(p) e^{-i\tilde{\phi}_1}}{i(\tilde{\phi}_1' - \tilde{\phi}_2')} . \tag{44}$$

If the right side of Equation (44) is now averaged over a period of $\phi_2 - \phi_1$, and the result divided into real and imaginary parts,

$$K_{10}' = 0 \tag{45}$$

$$\tilde{\phi}_{10}' = M_2 \left[\frac{K_{10}^{\;2} + 2K_{20}^{\;2}}{\tilde{\phi}_2' - \tilde{\phi}_1'} \right] \tag{46}$$

By symmetry,

$$K_{20}' = 0 \tag{46}$$

$$\tilde{\phi}_{20}' = M_2 \left[\frac{K_{20}^{\;2} + 2K_{10}^{\;2}}{\tilde{\phi}_1' - \tilde{\phi}_2'} \right] \tag{47}$$

Equations (45 - 47) show that the first approximation to the solution of Equation (30) is an epicycle with no damping but with frequencies which differ from those of the solution of the linearized equation. The corrections to the frequencies are the $\tilde{\phi}_{10}'$ s given by Equations (46) and (48).

$$\therefore \quad \phi_1' = \tilde{\phi}_1' - M_2 \left[\frac{K_{10}^{\;2} + 2K_{20}^{\;2}}{\tilde{\phi}_1' - \tilde{\phi}_2'} \right] \tag{48}$$

$$\phi_2' = \tilde{\phi}_2' + M_2 \left[\frac{K_{20}^2 + 2K_{10}^2}{\tilde{\phi}_1' - \tilde{\phi}_2'} \right] \tag{49}$$

In order to compare these results with Equations (37 - 38), Equation (48) and (49) are first added and then multiplied. After making use of Equations (32) and (33), the following relations may be obtained.

$$\phi_1' + \phi_2' = \bar{\nu} + M_2 \left(\frac{K_{10}^2 - K_{20}^2}{\tilde{\phi}_1' - \tilde{\phi}_2'} \right) \tag{50}$$

$$\phi_1' \cdot \phi_2' = M_0 + M_2 \left[K_{10}^2 + K_{20}^2 + \frac{K_{10}^2 \tilde{\phi}_1' - K_{20}^2 \tilde{\phi}_2'}{\tilde{\phi}_1' - \tilde{\phi}_2'} \right]$$

$$- (M_2)^2 \left[\frac{(K_{10}^2 + 2K_{20}^2)(K_{20}^2 + 2K_{10}^2)}{(\tilde{\phi}_1' - \tilde{\phi}_2')^2} \right] \tag{51}$$

But if second order quantities are neglected, Equations (50 - 51) are the same as Equations (37 - 38). Thus for twice the work we get the same result.

As a further basis for our quasi-linearization technique a special form of Equation (30) will be considered for which an exact solution is known. This case is that of a statically stable non-spinning missile in planar yawing motion. If λ is replaced by $\delta e^{i\theta}$ where θ is the orientation angle of the plane of yaw, Equation (30) assumes the form

$$\delta'' - (M_0 + M_2 \delta^2)\delta = 0 \tag{52}$$

where $M_0 < 0$. For planar yawing motion δ must go through zero and, hence, the two amplitudes must be equal ($K_{10} = K_{20} = K$). Finally according to Equation (37) the frequencies for a non-spinning missile differ only in sign, ($\phi_1' = -\phi_2' = \phi$). The quasi-linear solution of Equation (52) is a special form of an epicycle,

$$\begin{aligned} \delta &= K\,(e^{i\phi_1} + e^{i\phi_2})e^{-i\theta} \\ &= \delta_m \cos(\phi' p + \phi_0), \end{aligned} \tag{53}$$

for which Equation (38) reduces to

$$\phi^2 = - M_0 - \frac{3}{4} \delta_0^{\ 2} M_2 \tag{54}$$

where $\delta_m \doteq 2K$. (maximum value of δ)

$$\phi_0 = \phi_{10} - \theta = \theta - \phi_{20}.$$

The exact solution of Equation (52) is an elliptic function with period determined by complete elliptic integral of the first type. Thus by use of a little algebra and a table of complete elliptic integrals it is possible to compare the periods predicted by Equation (54) with the period of the exact solution of Equation (52). In doing this two cases must be considered:

1. A moment which grows faster than a linear moment ($M_2 < 0$)
2. A moment which grows slower than a linear moment ($M_2 > 0$) and, therefore, actually changes sign for $\left|\frac{M_2 \delta^2}{M_0}\right| = 1$.

Although a quasi-linear approximation may be reasonably good for reasonably large values of $\left|\frac{M_2 \delta^2}{M_0}\right|$ when M_2 is negative, it certainly cannot be good for values of $\left|\frac{M_2 \delta^2}{M_0}\right|$ near unity when M_2 is positive.

With this in mind we may now state the surprising results of this comparison with the exact theory. For negative M_2's, Equation (82) predicts the period with less than 1% error whenever the non-linear moment contribution is less than five times the linear moment $\left(\left|\frac{M_2 \delta^2}{M_0}\right| < 5\right)$. For positive M_2's the error will be less than a percent when the non-linear moment contribution is less than one-half the linear moment $\left(\left|\frac{M_2 \delta^2}{M_0}\right| < \frac{1}{2}\right)$. In this case the error, however, rises quite rapidly for larger angles. As a result of these facts it is reasonable to make use of Equations (35 - 38) with considerable optimism.

Turning now to the question of more general non-linearities an examination of the algebra used in Equations (35 - 38) indicates that any polynomial function of δ^2 could be used. In order to obtain the effective values of higher powers of δ^2 we look for those terms in δ^{2n} which upon multiplication by λ yield terms in $K_{10} e^{i\phi_1}$ or $K_{20} e^{i\phi_2}$. Clearly the only terms which have this property are constants, $e^{i(\phi_2 - \phi_1)}$, and $e^{i(\phi_1 - \phi_2)}$. In the binomial expansion of δ^{2n} the only term in which these exponentials appear are various multiples of the cosine, i.e. $CK_{10}K_{20}\left[e^{i(\phi_1 - \phi_2)} + e^{i(\phi_2 - \phi_1)}\right]$. Although the constant terms are unaltered in all three forms of effective yaws, $\left[\delta^{2n}\right]_{e1}$, $\left[\delta^{2n}\right]_{e2}$ and $\left[\delta^{2n}\right]_{e}$, the cosine term makes different contributions to these forms. For $\left[\delta^{2n}\right]_{e1}$ its contribution is CK_{20}^2, for $\left[\delta^{2n}\right]_{e2}$ it is CK_{10}^2, and for $\left[\delta^{2n}\right]_{e}$ it is $C\left(\frac{K_{10}^2 \phi_1' - K_{20}^2 \phi_2'}{\phi_1' - \phi_2'}\right)$.

As an example of this algorithm we calculate the three effective forms of δ^4.

From Equation (25),

$$\delta^4 = \left[a + b(e^{i(\phi_1 - \phi_2)} + e^{i(\phi_2 - \phi_1)})\right]^2 \tag{55}$$

where $a = K_{10}^2 + K_{20}^2$

$$b = K_{10}K_{20}$$

The exponential form of the cosine allows an easy selection of the constant and cosine terms of the expansion. These particular terms of the expansion will be identified by brackets on δ^4

$$\left[\delta^4\right] = a^2 + 2b^2 + 2a\left[\quad\right] \tag{56}$$

where $\left[\quad\right] = b(e^{i(\phi_1 - \phi_2)} + e^{i(\phi_2 - \phi_1)})$

In order to obtain $\left[\delta^4\right]_{e1}$, $\left[\delta^4\right]_{e2}$, or $\left[\delta^4\right]_{e}$, K_{20}^2, K_{10}^2, or $\frac{K_{10}^2 \phi_1' - K_{20}^2 \phi_2'}{\phi_1' - \phi_2'}$ respectively should be inserted in the empty brackets of Equation (56).

$$\therefore \quad \left[\delta^4\right]_{e1} = a^2 + 2b^2 + 2a \left[K_{20}^{\ 2}\right] \tag{57}$$

$$\left[\delta^4\right]_{e2} = a^2 + 2b^2 + 2a \left[K_{10}^{\ 2}\right] \tag{58}$$

$$\left[\delta^4\right]_{e} = a^2 + 2b^2 + 2a \left[\frac{K_{10}^{\ 2}\,\phi_1' - K_{20}^{\ 2}\,\phi_2}{\phi_1' - \phi_2'}\right] \tag{59}$$

In Table I values of δ^{2n} are given for all values of n between one and eight.

4. GEOMETRIC NON-LINEARITIES OF LARGE YAWING MOTION

The basic quantities measured in free flight are the co-ordinates of missile's center of mass and the direction cosines of its axis of symmetry. Since the usual formulas[5] relating the direction cosines and the motion of the c.m. to the complex yaw λ were derived for small yaws, the exact relations have to be derived for this report. In this derivation we will find it convenient to keep the 2-axis in the horizontal plane (fixed-plane co-ordinates). Unfortunately these co-ordinates are not the same as the non-rolling co-ordinates ($\hat{\nu} = 0$) and it is, therefore, necessary to calculate $\hat{\nu}$ for these co-ordinates. Surprisingly enough in Reference 8 it is shown that not only is $\hat{\nu}$ finite but it has a non-zero average value. In our development the cumbersome Eulerian angles of that report will not be used and the desired results will be obtained in a somewhat simpler fashion.

Co-ordinates in the free flight range system will be identified by (x_1, x_2, x_3) and in the fixed-plane system by (y_1, y_2, y_3). The range system has a 1-axis pointed downrange along the intersection of horizontal plane and vertical plane containing the gun. The 2-axis lies in the horizontal plane pointing to the left and the 3-axis up. The fixed-plane co-ordinates have the 1-axis along the missile's axis, the 2-axis in the horizontal plane pointing to the right, and the 3-axis down. Finally the non-spinning co-ordinates, which are our fundamental co-ordinates in the theory, have the 1-axis along the missile's axis, the 2-axis initially pointing to the right in the horizontal plane but moving so that $\hat{\nu}$ is zero and the 3-axis fixed by the right-hand rule.

TABLE I

$$\left[\delta^{2}\right] = a + \left[\quad\right]$$

$$\left[\delta^{4}\right] = a^{2} + 2b^{2} + 2a\left[\quad\right]$$

$$\left[\delta^{6}\right] = a^{3} + 6ab^{2} + 3(a^{2} + b^{2})\left[\quad\right]$$

$$\left[\delta^{8}\right] = a^{4} + 12a^{2}b^{2} + 6b^{4} + 4(a^{3} + 3ab^{2})\left[\quad\right]$$

$$\left[\delta^{10}\right] = a^{5} + 20a^{3}b^{2} + 30ab^{4} + 5(a^{4} + 6a^{2}b^{2} + 2b^{4})\left[\quad\right]$$

$$\left[\delta^{12}\right] = a^{6} + 30a^{4}b^{2} + 90a^{2}b^{4} + 20b^{6} + 6(a^{5} + 10a^{3}b^{2} + 10ab^{4})\left[\quad\right]$$

$$\left[\delta^{14}\right] = a^{7} + 42a^{5}b^{2} + 210a^{3}b^{4} + 140ab^{6} + 7(a^{6} + 15a^{4}b^{2} + 30a^{2}b^{4} + 5b^{6})\left[\quad\right]$$

$$\left[\delta^{16}\right] = a^{8} + 56a^{6}b^{2} + 420a^{4}b^{4} + 560a^{2}b^{6} + 70b^{8} + 8(a^{7} + 21a^{5}b^{2} + 70a^{3}b^{4} + 35ab^{6})\left[\quad\right]$$

In order to calculate the complex yaw it is necessary to know the components in the range system of the unit vectors, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, along the fixed-plane axes. Since the fixed plane 1-axis lies along the missile's axis of symmetry, components of the unit vector $\vec{e}$ are (n_1, n_2, n_3) where the n_i are direction cosines of the missile's axis with respect to the range system. The restriction of the fixed plane 2-axis to the horizontal plane is equivalent to the requirement that the third component of $\vec{e}_2$ be zero. This together with the requirement that it be perpendicular to the 1-axis and pointing to the right completely determine the components of the unit vector along the 2-axis.

$$\therefore \quad \vec{e}_2 = \left(\frac{n_2}{\sqrt{n_1^{\,2} + n_2^{\,2}}}, \frac{-n_1}{\sqrt{n_1^{\,2} + n_2^{\,2}}}, 0 \right) \tag{61}$$

According to the right hand rule, the third unit vector is equal to the cross product of the first two.

$$\vec{e}_3 = \vec{e}_1 \times \vec{e}_2 = (n_1, n_2, n_3) \times \left(\frac{n_2}{\sqrt{n_1^{\,2} + n_2^{\,2}}}, \frac{-n_1}{\sqrt{n_1^{\,2} + n_2^{\,2}}}, 0 \right)$$

$$= \left(\frac{n_1 n_3}{\sqrt{n_1^{\,2} + n_2^{\,2}}}, \frac{n_2 n_3}{\sqrt{n_1^{\,2} + n_2^{\,2}}}, -\sqrt{n_1^{\,2} + n_2^{\,2}} \right) \tag{62}$$

With this information the matrix equation for changing from the range coordinates to fixed-plane co-ordinates can now be written:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} n_1 & n_2 & n_3 \\ \frac{n_2}{\sqrt{n_1^{\,2} + n_2^{\,2}}} & \frac{-n_1}{\sqrt{n_1^{\,2} + n_2^{\,2}}} & 0 \\ \frac{n_1 n_3}{\sqrt{n_1^{\,2} + n_2^{\,2}}} & \frac{n_2 n_3}{\sqrt{n_1^{\,2} + n_2^{\,2}}} & -\sqrt{n_1^{\,2} + n_2^{\,2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{63}$$

If the vector $(\dot{x}_1, \dot{x}_2, \dot{x}_3)$ is inserted on the right side of Equation (63), the left side would yield the components of the velocity vector in the fixed-plane co-ordinates. These co-ordinates will be denoted by the symbols $(u_1, \hat{u}_2, \hat{u}_3)$ and the symbols without the circumflex will be reserved for the non-rolling co-ordinate system.

$$\therefore \quad \hat{u}_2 = \frac{n_2\dot{x}_1 - n_1\dot{x}_2}{\sqrt{n_1^2 + n_2^2}} \tag{64}$$

$$\hat{u}_3 = \frac{n_1n_3\dot{x}_1 + n_2n_3\dot{x}_2 - (n_1^2 + n_2^2)\dot{x}_3}{\sqrt{n_1^2 + n_2^2}} \tag{65}$$

Dividing by the magnitude of the velocity vector, $u = \dot{x}_1 \left[1 + \left(\frac{dx_2}{dx_1}\right)^2 + \left(\frac{dx_3}{dx_1}\right)^2\right]^{1/2}$, and simplifying, we can obtain the following expressions for the components of the complex yaw in the fixed-plane system:

$$\hat{\lambda}_2 = \frac{\hat{u}_2}{u} = \frac{n_2 - n_1 \frac{dx_2}{dx_1}}{(\sqrt{n_1^2 + n_2^2})\sqrt{1 + (\frac{dx_2}{dx_1})^2 + (\frac{dx_3}{dx_1})^2}} \tag{66}$$

$$\hat{\lambda}_3 = \frac{\hat{u}_3}{u} = \frac{n_1n_3 + n_2n_3(\frac{dx_2}{dx_1}) - (n_1^2 + n_2^2)(\frac{dx_3}{dx_1})}{(\sqrt{n_1^2 + n_2^2})\sqrt{1 + (\frac{dx_2}{dx_1})^2 + (\frac{dx_3}{dx_1})^2}} \tag{67}$$

For the case of a flat trajectory, the derivatives are the same order of magnitude as n_2 and n_3 or smaller and, if we make use of the fact that $n_1^2 = 1 - n_2^2 - n_3^2$, we can obtain the first order approximation of the yaw components

$$\begin{aligned} \hat{\lambda}_2 &= n_2 - \frac{dx_2}{dx_1} \\ \hat{\lambda}_3 &= n_3 - \frac{dx_3}{dx_1} \end{aligned} \tag{68}$$

Turning now to the problem of computing the axial angular velocity of our fixed-plane co-ordinates, we first must state the definition of the angular velocity vector in terms of unit vectors along these co-ordinate axes $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$.

$$\begin{pmatrix} \Omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\vec{e}}_2 \cdot \vec{e}_3 \\ \dot{\vec{e}}_3 \cdot \vec{e}_1 \\ \dot{\vec{e}}_1 \cdot \vec{e}_2 \end{pmatrix} = - \begin{pmatrix} \vec{e}_2 \cdot \dot{\vec{e}}_3 \\ \vec{e}_3 \cdot \dot{\vec{e}}_1 \\ \vec{e}_1 \cdot \dot{\vec{e}}_2 \end{pmatrix} \tag{69}$$

or $$\hat{\nu} = \frac{\Omega_1 d}{u} = \vec{e}_2{}' \cdot \vec{e}_3 \tag{70}$$

$$\hat{\mu} = \frac{(\omega_2 + i\omega_3)d}{u} = \vec{e}_1{}' \cdot (-\vec{e}_3 + i\,\vec{e}_2) \tag{71}$$

where $(\quad)'$ denotes $\frac{d}{dp}$.

Substituting in Equation (70) the co-ordinates of the vectors $\vec{e}_2$ and $\vec{e}_3$ from Equations (61-62) we have

$$\hat{\nu} = \begin{pmatrix} \frac{n_2}{\sqrt{1 - n_3^2}} \\ \frac{-n_1}{\sqrt{1 - n_3^2}} \\ 0 \end{pmatrix}' \cdot \begin{pmatrix} \frac{n_1 n_3}{\sqrt{1 - n_3^2}} \\ \frac{n_2 n_3}{\sqrt{1 - n_3^2}} \\ -\sqrt{1 - n_3^2} \end{pmatrix} = \frac{n_3\,(n_1 n_2{}' - n_2 n_1{}')}{1 - n_3^2} \tag{72}$$

Similarly,

$$\hat{\mu} = \hat{\mu}_2 + i\hat{\mu}_3$$

$$= \frac{n_3'}{\sqrt{1 - n_3^2}} + i\left(\frac{n_1{}' n_2 - n_2{}' n_1}{\sqrt{1 - n_3^2}}\right) \tag{73}$$

$$\therefore \quad \hat{\nu} = - \hat{\mu}_3 \left(\frac{n_3}{-\sqrt{1 - n_3^2}} \right) \tag{74}$$

If the angle between the missile's axis and a vertical axis is denoted by ψ, then $n_3 = \cos \psi$ and Equation (74) reduces to

$$\hat{\nu} = - \hat{\mu}_3 \cot \psi \tag{75}$$

Since $\hat{\mu}_2$ is the angular velocity of the missile's axis about the 2-axis which is fixed in the horizontal plane, its integral should be related to ψ. Integrating $\hat{\mu}_2$ as given by Equation (73), we see that this is true.

$$\int_0^p - \hat{\mu}_2 \, dp = \arccos n_3(p) - \arccos n_3(0) \tag{76}$$

$$\therefore \quad \psi = \int_0^p - \hat{\mu}_2 \, dp + \psi_0 \tag{77}$$

Although Equations (75) and (77) are very useful for the numerical integration of the complete equations of motion, we will find Equation (72) to be much more convenient for the purposes of this report.

If the flat trajectory approximation $x_1 = pd$ is used, a first order approximation of $\hat{\nu}$ may now be obtained from Equations (67), (68), and (72).

$$\begin{aligned} \hat{\nu} &= n_3 n_2' = (\hat{\lambda}_3 + \frac{x_3'}{d}) \, (\hat{\lambda}_2' + \frac{x_2''}{d}) \\ &= \hat{\lambda}_3 \hat{\lambda}_2' + T \end{aligned} \tag{78}$$

where $T = \frac{x_3'}{d} \hat{\lambda}_2' + \frac{x_2''}{d} \hat{\lambda}_3 + \frac{x_3' \, x_2''}{d^2}$.

A good approximation to the yaw components is an epicycle.

$$\hat{\lambda}_2 = K_{10} \cos \hat{\phi}_1 + K_{20} \cos \hat{\phi}_2 \tag{79}$$

$$\hat{\lambda}_3 = K_{10} \sin \hat{\phi}_1 + K_{20} \sin \hat{\phi}_2 \tag{80}$$

where K_{i0} are constants and

$\hat{\phi}_i$ are linear functions* of p.

$$\therefore \hat{\lambda}_3 \hat{\lambda}_2' = (K_{10} \sin \hat{\phi}_1 + K_{20} \sin \hat{\phi}_2)(- K_{10} \hat{\phi}_1' \sin \hat{\phi}_1 - K_{20} \hat{\phi}_2' \sin \hat{\phi}_2)$$

$$= - \frac{1}{2} \left[K_{10}^2 \hat{\phi}_1' + K_{20}^2 \hat{\phi}_2' \right] + \frac{1}{2} \left[K_{10}^2 \hat{\phi}_1' \cos 2\hat{\phi}_1 + K_{20}^2 \hat{\phi}_2' \cos 2\hat{\phi}_2 \right]$$

$$- (\hat{\phi}_1' + \hat{\phi}_2') K_{10} K_{20} \sin \hat{\phi}_1 \sin \hat{\phi}_2 \qquad (81)$$

From Equation (81) it can be seen that $\hat{v}$ has a non-zero average value of $- \frac{1}{2} \left[K_{10}^2 \hat{\phi}_1' + K_{20}^2 \hat{\phi}_2' \right]$. (Fortunately, for small yaws, this can be neglected.)

Turning to the small trajectory term T we consider only the effect of gravity and lift force and obtain the following relations for constant velocity u_0 from Reference 5.

$$\frac{x_2'}{d} = Q_1 + J_L \left[\frac{K_{10}}{\hat{\phi}_1'} \sin \hat{\phi}_1 + \frac{K_{20}}{\hat{\phi}_2'} \sin \hat{\phi}_2 \right] \qquad (82)$$

$$\frac{x_3'}{d} = Q_2 - \frac{gd}{u_0^2} p - J_L \left[\frac{K_{10}}{\hat{\phi}_1'} \cos \hat{\phi}_1 + \frac{K_{20}}{\hat{\phi}_2'} \cos \hat{\phi}_2 \right] \qquad (83)$$

where Q_i are small constants**, and

g is the acceleration due to gravity

$J_L = \frac{\rho d^3}{m} K_L$

A little algebraic manipulation results in the relation that

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* The circumflex on the $\hat{\phi}_i$'s is to indicate that these angles are measured in the fixed plane system and not in the non-rolling system of Section 2.

** This is due to the assumption of a flat trajectory.

$$T = -\left(\frac{1}{2}\right) \quad J_L{}^2 \left[\frac{K_{10}{}^2}{\phi_1{}^{\prime}} + \frac{K_{20}{}^2}{\phi_2{}^{\prime}}\right] + \text{fluctuating terms} \qquad (84)$$

Since for most missiles $\frac{J_L{}^2}{\hat{\phi}_1{}^{\prime}} \sim 10^{-6}$, T is effectively the same order as $K_{10}{}^4$ and we need only consider the fourth order terms which are independent of the trajectory to obtain a complete fourth order approximation to the non-fluctuating or "D. C. component" of ν. This expression is calculated in Appendix B.* If we add T to this and define $[\hat{\nu}]$ to be the "D. C. component" of $\hat{\nu}$, we see that

$$[\hat{\nu}] = -\frac{1}{2} \left[K_{10}{}^2 \ \hat{\phi}_1{}^{\prime} \left(1 + \frac{K_{10}{}^2 + 2\,K_{20}{}^2}{4}\right) + K_{20}{}^2 \ \hat{\phi}_2{}^{\prime} \left(1 + \frac{K_{20}{}^2 + 2\,K_{10}{}^2}{4}\right)\right]$$
$$- \left(\frac{1}{2}\right) J_L{}^2 \left(\frac{K_{10}{}^2}{\hat{\phi}_1{}^{\prime}} + \frac{K_{20}{}^2}{\hat{\phi}_2{}^{\prime\prime}}\right) + \ldots . \qquad (85)$$

With these results we can now calculate either the exact position or the average position of 2 and 3-axes of the non-rolling co-ordinates. Since our fixed plane co-ordinates are turning at an angular velocity of $\hat{\nu}$ with respect to the non-rolling co-ordinates, we have that

$$\lambda = \hat{\lambda}\ e^{i\int_0^p \hat{\nu}\, dp} \qquad (86)$$

As it shall be shown later, $\hat{\nu}$, in Equation (86) may be replaced by $[\hat{\nu}]$ for the data reduction of large yaw firings.

The effect of large yawing motions on the results of Section 2 manifests it self in two ways. First the distinction between the non-spinning co-ordinates and the fixed-plane co-ordinates becomes important. According to Equations (85-86) these co-ordinate systems rotate with respect to each other with a non-z

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* In Appendix B the frequency for pure precession is compared with the approximate relation based on this expression for $[\hat{\nu}]$. As can be seen in Fig. 15, the agreement is good for $\delta = \sin 45^{o}$.

average$\left[\hat{\nu}\right]$. If $\hat{\phi}_1'$ and $\hat{\phi}_2'$ denote the epicycle frequencies for the range fixed-plane co-ordinates, it can be seen from Equations (85) and (86) and the assumption that the motion in fixed plane coordinates is also epicyclic that*

$$\phi_1' = \hat{\phi}_1' + \left[\hat{\nu}\right] \doteq \hat{\phi}_1' - \frac{1}{2}\left[K_{10}^2\hat{\phi}_1' + K_{20}^2\hat{\phi}_2'\right] \tag{87}$$

$$\phi_2' = \hat{\phi}_2' + \left[\hat{\nu}\right] \doteq \hat{\phi}_2' - \frac{1}{2}\left[K_{10}^2\,\hat{\phi}_1' + K_{20}^2\,\hat{\phi}_2'\right]. \tag{88}$$

The second effect which is the appearance of the cosine of the yaw angle, may be seen in the equation of undamped yawing motion in the non-rolling system. This may be obtained from Equation (20) as

$$\lambda'' - \left(\frac{\ell'}{\ell} + i\bar{\nu}\right)\lambda' - M\lambda = 0 \tag{89}$$

where $M = \ell\, k_2^{-2}\; J_M \doteq M_0 + M_2^*\;\; \delta^2$

$$M_0 = k_2^{-2}\; J_{M_0}$$

$$M_2^* = M_2 - \frac{1}{2}M_0 = k_2^{-2}\left(J_{M_\delta 2} - \frac{1}{2}J_{M_0}\right)$$

Equation (89) differs from Equation (30) by the $\frac{\ell'}{\ell}\lambda'$ term and the presence of the cosine term in M. In order to obtain the correct form of Equations (37-38) we must derive a good approximation to $\frac{\ell'}{\ell}$.

$$\frac{\ell'}{\ell} = \left(\frac{1}{2}\right)\frac{(\ell^2)'}{\ell^2}$$

$$= -\left(\frac{1}{2}\right)\frac{(\delta^2)'}{(1-\delta^2)} \tag{90}$$

---

*For rapidly spinning models at high yaw Equation (88) can impose a large correction on ϕ_2'. If $\hat{\phi}_1'/\hat{\phi}_2' = 40$ and $K_1 > .23$, ϕ_2' becomes negative and the product of the rates in the non-rolling system would be negative.

$$\doteq -\left(\frac{1}{2}\right)\left(\delta^2\right)'$$

Inserting Equation (25) in Equation (90) we have

$$\frac{\ell'}{\ell} = -\frac{i}{2} K_{10} K_{20} \left[(\phi_1' - \phi_2') e^{i(\phi_1 - \phi_2)} + (\phi_2' - \phi_1') e^{i(\phi_2 - \phi_1)} \right] \tag{91}$$

Substituting Equation (91) in Equation (89), and operating on Equation (89) in the same way as we treated Equation (30), the following two equations may be obtained from the coefficients of $K_{10}e^{i\phi_1}$ and $K_{20}e^{i\phi_2}$ respectively

$$\phi_1'^2 - \bar{\nu}\phi_1' + M_0 + M_2^* \delta_{e1}^2 + K_{20}^2 (\phi_1' - \phi_2')\left(\frac{\phi_2'}{2}\right) = 0 \tag{92}$$

$$\phi_2'^2 - \bar{\nu}\phi_2' + M_0 + M_2^* \delta_{e2}^2 + K_{10}^2 (\phi_2' - \phi_1')\left(\frac{\phi_1'}{2}\right) = 0 \tag{93}$$

Eliminating first M_0 and then $\bar{\nu}$ between Equations (92) and (93),

$$\phi_1' + \phi_2' + \frac{1}{2}\left[K_{10}^2 \phi_1' + K_{20}^2 \phi_2' \right] = \bar{\nu} + M_2^* \left(\frac{K_{10}^2 - K_{20}^2}{\phi_1' - \phi_2'} \right) \tag{94}$$

$$\phi_1' \cdot \phi_2' + \frac{1}{2}\left[K_{10}^2 \phi_1'^2 + K_{20}^2 \phi_2'^2 \right] = M_0 + M_2^* \delta_e^2 \tag{95}$$

Thus the geometric non-linearity of Equation (89) introduce correction terms to the left sides of Equations (37 - 38) of the previous section and replaces M_2 by $M_2^* = M_2 - \frac{1}{2} M_0$ in those equations. If the frequencies in Equations (94 - 95) are replaced by their fixed plane values from Equation (87 - 88), and fourth powers of K_{10}'s are neglected, Equations (94 - 95) may be written in the following useful form.

$$\hat{\phi}_1' + \hat{\phi}_2' - \frac{1}{2}\left[K_{10}^2 \hat{\phi}_1' + K_{20}^2 \hat{\phi}_2' \right] = \bar{\nu} + M_2^* \left(\frac{K_{10}^2 - K_{20}^2}{\phi_1' - \phi_2'} \right) \tag{96}$$

$$\hat{\phi}_1' \cdot \hat{\phi}_2' \left[1 - \frac{1}{2}\left(K_{10}^2 + K_{20}^2\right) \right] \doteq M_0 + M_2^* \delta_e^2 \tag{97}$$

An examination of Equation (97) shows that the effect of a large percentage change in the frequencies implied by Equations (87 - 88) is almost completely cancelled when the terms in the cosine of yaw are properly handled. As a check on the derivation of Equations (96 - 97) they are derived in Appendix C directly from the yaw equation expressed in the fixed-plane system. This derivation shows that the assumption implied in Equations (87 - 88) is equivalent to the basic assumption underlying this whole report.

5. EXPERIMENTAL RESULTS

In Section 3, a very simple example of non-linear yawing motion with two degrees of freedom is considered in great detail. The very same quasi-linear assumption may be applied to a much more complicated system. In appendices D and E the yawing and swerving motion of a missile for which all aerodynamic coefficients are quadratic functions of δ is considered. Equations relating the parameters of the epicycle approximation (frequencies and damping exponents) to the coefficients of the parent non-linear equation and the amplitudes of both modes are derived. Since these relations are essentially approximations, their value must be determined by either numerical or experimental checks.

A good check on these approximations can be obtained by means of the actual free flight motion of a missile acted on by non-linear forces and moments. For this reason various results of firing tests on BRL's spark ranges[9] were examined for possible verification of the quasi-linear hypothesis. The results of this investigation were quite encouraging.

The effect of a cubic static moment on three different programs was first considered. These programs were a finned missile program (81mm mortars[10]), a body of revolution program (Army-Navy Spinner Rocket program[6]), and a large yaw body of revolution program fired by E. Roecker.[13])

In the finned missile program a free flight range value of K_M was computed from the product of the frequencies. According to Equation (38a) and the definitions of M_i

$$K_{M_{range}} \doteq \frac{\rho d^5}{B} \phi_1 \cdot \phi_2 = K_{M_0} + K_{M_{\delta^2}} \delta_e^{\,2} \qquad (98)$$

Thus the range value of K_M, as obtained from an epicycle fit of individual rounds, should be a linear function of the corresponding effective squared yaws. In Figure 2 the range values of K_M are plotted against δ_e^2 and different lines for each configuration tested are drawn through corresponding points.* The linearity exhibited by these experimental points constitutes the first check of the theory.

It should be emphasized that in all cases, epicycles with damping were fitted to the data and although the exponentially varying K_1's actually should have appeared in the calculation of δ_e^2 they were approximated by their midrange values K_{10}'s. The approximation is implied by our basic quasi-linear assumption.

The Army-Navy Spinner Rocket program consisted of three model lengths each with three different center of mass positions and fired at three Mach numbers. A maximum of twenty-seven values of $K_{M_{\delta^2}}$ was therefore possible. Since for each case at least four points with a reasonable spread in δ_e^2 are necessary for a good determination of $K_{M_{\delta^2}}$, it was actually possible to obtain only sixteen values from a careful analysis of the 126 record rounds of the program.

These values with their standard errors are listed in Table II. A sample plot of the experimental points for the 9-caliber model at Mach number of 1.8 is given in Figure 3.

For some of the Mach numbers and configurations, wind tunnel data taken by R. Krieger were available. The data for the overturning moment were fitted by cubics and the cubic coefficients $(K_{M_{\delta^2}})$ are listed in Table II. At all seven points of comparison the agreement is good.

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* Since these finned missiles were not spinning, $\phi_1' = -\phi_2'$ and the effective squared yaw assumes the concise form $\frac{3}{2}(K_{10}^2 + K_{20}^2.)$

In Reference 10, the $K_{M_{range}}$'s were erroneously plotted against $\overline{\delta^2} \doteq K_{10}^2 + K_{20}^2$ and, hence, the slopes obtained in that report are actually $\frac{3}{2} K_{M_{\delta^2}}$.

TABLE II

$K_{M_{\delta}2}$ FOR THE ARMY-NAVY SPINNER ROCKET

Mach Number	Forward Center of Mass		Middle Center of Mass		Rear Center of Mass	
	Range	Wind Tunnel	Range	Wind Tunnel	Range	Wind Tunnel
			Five Caliber Long Models			
1.3		- 2	- 3 ± 1	0	1 ± 2	3
1.8		-14		- 7	3 ± 1	0
			Seven Caliber Long Models			
1.3	- 21 ± 1		- 5 ± 1		7 ± 2	5
1.8	- 12 ± 1	-11		- 3	0 ± 3	3
2.5	- 28 ± 1	-26		- 11		
			Nine Caliber Long Models			
1.3	- 35 ± 1				.1 ± .01 *	
1.8	- 30 ± 4		- 16 ± 1		- 3.7 ± .3	
2.5			- 30 ± 2		- 1 ± .6	

* The center of mass for these RCM models is .2 cal. rear of its usual location.

Since models with three different center of mass locations were fired, a second check is possible. If the normal force is expanded as a cubic function of δ, the usual center of mass relations provide that

$$K_{M_0}(q) = K_{M_0} + q\, K_{N_0} \tag{99}$$

$$K_{M_{\delta^2}}(q) = K_{M_{\delta^2}} + q\, K_{N_{\delta^2}} \tag{100}$$

where $K_{M_i}(q)$ are the moment coefficients for a center of mass located q calibers rear of the center of mass for the K_{M_i}'s.

This means that the K_{M_0} and $K_{M_{\delta^2}}$'s for different center of mass locations are linear functions of location. It was possible to measure three values of $K_{M_{\delta^2}}$, for only one configuration and one Mach number. In Figure 4 both K_{M_0} and $K_{M_{\delta^2}}$ for this case, are plotted against center of mass location. The fact that the cubic coefficients as well as the linear coefficients fall on a straight line is another point in favour of the theory. The slopes of these lines are $K_{N_{\delta^2}}$ and K_{N_0} respectively.

For six other cases it was possible to compute $K_{N_{\delta^2}}$ from two values of $K_{M_{\delta^2}}$ and all seven values together with the corresponding linear values (K_{N_0}) are tabulated in Table III. Once again Krieger's wind tunnel data were analysed and it was found that at all points of comparison the wind tunnel results were in good agreement with center of mass values.

In Appendix D, it is shown that the lift force and Magnus force coefficients may be directly measured from the swerving motion when it is large enough. According to this appendix, most of the swerve is associated with the lower frequency and, hence, the range values of K_L and K_F for each round should be plotted against the corresponding value of $\delta_{e2}^{\,2}$. A sample plot of this process is given in Figure 5.

TABLE III

Mach Number	K_{N_0}			$K_{N_\delta 2}$		
	Center of Mass	Swerve	Wind Tunnel	Center of Mass	Swerve	Wind Tunnel
		Five Caliber Models				
1.3	.98	.99	.92	7	5.2 ± .2	5
1.8	1.13	1.13	1.10	--	10 ± 2	7
		Seven Caliber Models				
1.3	1.02	.98	--	20	10 ± 1	--
1.8	1.13	1.13	1.08	12	9 ± 4	10
2.5	1.21	1.20	1.21	18	27 ± 4	18
		Nine Caliber Models				
1.3	1.06	1.07		17	17 ± 1	
1.8	1.14	1.23		14.2 ± .4	15 ± 2	
2.5	1.31	1.20		34	30 ± 2	

In order to calculate $K_{N_{\delta^2}}$ from $K_{L_{\delta^2}}$, it is necessary to expand Equation (17) in powers of δ^2 and compare coefficients

$$\therefore \quad K_{N_0} = K_{L_0} + K_{D_0} \tag{101}$$

$$K_{N_{\delta^2}} = K_{L_{\delta^2}} - \frac{1}{2} K_{L_0} + K_{D_{\delta^2}} \tag{102}$$

Using these relations it was possible to calculate eight values of both K_{N_0} and $K_{N_{\delta^2}}$ from the swerving motion and these are given in Table III. With the exception of two values of $K_{N_{\delta^2}}$ for the 7-caliber models the agreement is remarkable. The reason for these two discrepancies is at present unknown.

Although the lateral displacement due to Magnus force is about one tenth that due to lift, it was possible to make two measurements of $K_{F_{\delta^2}}$. The experimental points for these two cases are given in Figure 6. The excellent internal agreement of these quite delicate measurements is extremely gratifying.

Finally in Roecker's large yaw program it was possible to check the treatment for the geometric non-linearities. In Figure 7 $K_{M_{range}}$, as calculated from Equation (97), is plotted against δ_e^2. Here we see that the data is essentially bilinear. Each line corresponds to a cubic segment in the moment plane. If the parameters of each cubic are calculated from the slope and intercept of its corresponding line in the $K_M - \delta_e^2$ plane, they can be pieced together to form a smooth moment plot. (Figure 8). (An examination of the spark shadowgraphs revealed the fact that flow separation occurs at about 21° and this explains the sudden change in the moment curve at this point.) In Figure 9, K_L is plotted versus δ_{e2}^2 and the corresponding lift force plot is given in Figure 10.

Since wind tunnel measurements for this configuration had been made by W. Buford, a comparison was possible. The data was divided into angles of yaw less than 21° and greater than 21° and pairs of cubics fitted. The resulting coefficients are compared in Table IV. The agreement is excellent.

As a result of this rather spectacular success with the non-linear static moment, the more difficult problem of Magnus and damping moments was then considered. In Appendix E it is shown that a quadratic dependence of K_T and $K_H - K_{MA}$ on δ so affects the damping of the epicycle that the usual linear formulas for $K_H - K_{MA}$ and K_T are actually the following combinations of coefficients:

$$(K_H - K_{MA})_{\text{range}} = K_{HO} - K_{MAO} + K^*_{H_\delta 2} \left(\frac{\phi_1' K_{20}^2 - \phi_2' K_{10}^2}{\phi_1' - \phi_2'}\right)$$
$$- K^*_{T_\delta 2} \left(\frac{B}{A}\right) \left(\frac{\phi_1' + \phi_2'}{\phi_1' - \phi_2'}\right) \left(K_{10}^2 - K_{20}^2\right) \tag{103}$$

$$K_{T_{\text{range}}} = K_{TO} + K^*_{T_\delta 2}\, \delta_e^2 + K^*_{H_\delta 2} \left(\frac{A}{B}\right) \left(\frac{K_{10}^2 \phi_1'^2 - K_{20}^2 \phi_2'^2}{\phi_1'^2 - \phi_2'^2}\right) \tag{104}$$

where $K^*_{H_\delta 2} = K_{H_\delta 2} - K_{MA_\delta 2} + \frac{1}{2} K_{MAO}$

$$K^*_{T_\delta 2} = K_{T_\delta 2} - \frac{1}{2} K_{T_0}.$$

Range values of $K_H - K_{MA}$ and K_T for the Army-Navy Spinner Rocket program were fitted by least squares to Equations (103-104) and values of K_{TO}, $K_{HO} - K_{MAO}$, $K_{T_\delta 2}$ and $K^*_{H_\delta 2}$ were computed. In all cases the coefficients of $K^*_{H_\delta 2}$ were small and this quantity was poorly determined. $K^*_{H_\delta 2}$ was therefore omitted from Equations (103-104) and they were fitted separately to range values of $K_H - K_{MA}$ and K_T.

Since this measurement depends on the damping exponents and hence is quite delicate, it was possible to obtain only eight values of $K_{T_\delta 2}$ from $K_{T_{\text{range}}}$ and four from $(K_H - K_{MA})_{\text{range}}$. These are listed in Table V

TABLE IV

Large Yaw Results for M = 2.3

	$K_{D_{\delta^2}}$	K_{M_0}	$K_{M_{\delta^2}}$	K_{L_0}	$K_{L_{\delta^2}}$	
Wind Tunnel	1.85	.825	- 1.1	1.01	4.0	before separation
Range	1.82	.829	- 1.0	0.99	3.6	$\delta \leqq \sin 21^{\circ}$
Wind Tunnel		.729*	0.0	1.45*	0.0	after separation
Range		.719*	0.1	1.43*	0.4	$\delta \geqq \sin 21^{\circ}$

* Evaluated at separation

TABLE V

Mach No.	$K_{F_{\delta^2}}$ C.M.	$K_{F_{\delta^2}}$ Swerve	$CP_{F_{\delta^2}}$		$K_{T_{\delta^2}}$ FCM	$K_{T_{\delta^2}}$ MCM	$K_{T_{\delta^2}}$ RCM
			Seven Caliber Long Models				
1.3	14	17 ± 1	4.4	$K_{T_{range}}$	- 16 ± 5	- 5 ± 1	--
				$(K_H - K_{MA})_{range}$	- 20 ± 14	--	--
1.8	22	--	4.8	$K_{T_{range}}$	- 34 ± 16	--	0 ± 1
				$(K_H - K_{MA})_{range}$	- 25 ± 14	--	--
2.5	19	--	5.1	$K_{T_{range}}$	- 36 ± 5	--	-6 ± 3
				$(K_H - K_{MA})_{range}$	- 28 ± 4	--	--
			Nine Caliber Long Models				
1.8	12	--*	6.3	$K_{T_{range}}$	- 29 ± 2	--	-5.2 ± .2
				$(K_H - K_{MA})_{range}$	- 26 + 3	--	--

* $K_{F_{\delta^2}}$ for Mach number 1.3 was given in Figure 6 as 8 ± 1.

NOL Wind Tunnel measurements were made by Luchuk and Sparks[11] for 7 caliber model at M between 1.6 and 2.5. According to these results $K_{F_{\delta^2}} \doteq 11$, $CP_{F_{\delta^2}} \doteq 5.4$.

and the agreement between pairs of $K_{T_{\delta^2}}$ is within the standard errors.*

In Figures 11 - 12, a pair of plots of experimental points for $K_{T_{range}}$ and $(K_H - K_{MA})_{range}$ are given.

In all four cases values of $K_{T_{\delta^2}}$ for two center of mass locations were obtained and $K_{F_{\delta^2}}$' s and $CP_{F_{\delta^2}}$' s were computed from these. At the Naval Ordnance Laboratory, Luchuk and Sparks[11] have made wind tunnel measurements of these quantities and their results, as given at the bottom of Table V, are in reasonable agreement with flight tests. Finally it should be noted in the Table that at one point of comparison of center of mass and swerve values of $K_{F_{\delta^2}}$, (the seven caliber models at M = 1.3) the agreement is good.

As a last example of this quasi-linear technique we will consider a body of revolution which displayed an extremely non-linear Magnus moment. The range values of K_T for this model were obtained by E. Roecker and are plotted versus δ_e^2 in Figure 13. Although these data are fitted by two lines, only the first line is well determined. In fact if the fourth point from the right were neglected, the large yaw values would be reasonably well represented by the dashed horizontal line.

In Figure 14 the corresponding cubic segments are plotted and compared with a BRL wind tunnel curve obtained by A. Platou. The good qualitative agreement for this strongly non-linear moment is remarkable.

Since the quasi-linear relations have been so well verified by experiment, they should be quite valuable in the important problem of the prediction of missile motion from a knowledge of the force and moment curves. This application of the quasi-linear technique is described in more detail in the next section.

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* Thus an observed dependence of $(K_H - K_{MA})_{range}$ on magnitude of yaw is not necessarily due to non-linear damping moments but may be due to a non-linear Magnus moment.

6. PREDICTION OF NON-LINEAR YAWING MOTION

Although the application of the theory of this report to range data analysis enhances markedly the value of spark ranges, an even more important application is the prediction of the yawing motion of a missile acted on by non-linear forces and moments. The experimental results of this report indicate that this inverse problem should be successfully handled by the same theory. In this section we will outline the procedure for this prediction problem.

First the sizes of the two arms must be obtained from the initial conditions

$$\therefore \quad \lambda_0 = K_1 e^{i\phi_1} + K_2 e^{i\phi_2} \tag{105}$$

$$\lambda_0' = (-\alpha_1 + i\phi_1')K_1 e^{i\phi_1} + (-\alpha_2 + i\phi_2')K_2 e^{i\phi_2} \tag{106}$$

In order to take care of the effect of damping the trajectory should be divided into intervals over which neither amplitude changes by more than 50%. The length of the first interval may be estimated from the linear damping. If the calculated quasi-linear damping is much larger, the interval may then be shortened and the process iterated. The lengths of the other intervals should be determined in a similar manner. Values of ϕ_i', frequencies in non-rolling co-ordinates, can now be computed from Equations (92 - 93) which are simple quadratic equations.* ($\hat{\phi}_1'$ may be computed from Equations (87 - 88).) Then values of α_i may be computed from Equations (E4 - E5) which are even simpler linear equations.*

* For ease of calculation the linearized values of ϕ_i's and α_is may be first placed in the coupling terms arising from the non-linear geometry and an iteration performed if needed.

If a more complex polynomial dependence of K_M and K_T on δ^2 is required, this may be handled in the following way.

Let

$$K_M = \sum_{k=0}^{n} K_{M_{\delta^{2k}}} \delta^{2k} \tag{107}$$

$$K_T = \sum_{k=0}^{n} K_{T_{\delta^{2k}}} \delta^{2k}. \tag{108}$$

Then Equations (92 - 93) and (E4 - E5) take on the following slightly more complex form:*

$$\phi_1'^2 - \bar{\nu}\phi_1' + \sum_{k=0}^{n} M_{2k}^{*} (\delta^{2k})_{e1} + K_{20}^2(\phi_1' - \phi_2')(\frac{\phi_2'}{2}) + \alpha_1(H - \alpha_1) = 0 \tag{109}$$

$$\phi_2'^2 - \bar{\nu}\phi_2' + \sum_{k=0}^{n} M_{2k}^{*} (\delta^{2k})_{e2} + K_{10}^2(\phi_2' - \phi_1')(\frac{\phi_1'}{2}) + \alpha_2(H - \alpha_2) = 0 \tag{110}$$

$$\alpha_1(2\phi_1' - \bar{\nu}) - H_0\phi_1' + \bar{\nu} \sum_{k=0}^{n} T_{2k}^{*}(\delta^{2k})_{e1} - H_{\delta^2}^{*} \left[(K_{10}^2 + K_{20}^2)\phi_1' + K_{20}^2\phi_2'\right]$$

$$- J_{L_{\delta^2}}(\phi_1' - \phi_2')K_{20}^2 + \phi_1'(\alpha_1 K_{10}^2 + \alpha_2 K_{20}^2) \tag{111}$$

$$+ (\frac{\alpha_1\phi_2' + \alpha_2\phi_1'}{2})K_{20}^2 = 0$$

$$\alpha_2(2\phi_2' - \bar{\nu}) - H_0\phi_2' + \bar{\nu} \sum_{k=0}^{n} T_{2k}^{*}(\delta^{2k})_{e2} - H_{\delta^2} \left[(K_{10}^2 + K_{20}^2)\phi_2' + K_{10}^2\phi_1'\right]$$

---

* Eqs. (92-93) are also slightly modified by the presence of damping.

$$- J_{L_{\delta}2} (\phi_2' - \phi_1')K_{10}^2 + \phi_2'(\alpha_1 K_{10}^2 + \alpha_2 K_{20}^2) \qquad (112)$$

$$+ \left(\frac{\alpha_1 \phi_2' + \alpha_2 \phi_1'}{2}\right)K_{10}^2 = 0$$

where $\sum_{k=0}^{n} M_{2k}^{*} \delta^{2k} = \frac{\rho d^5}{B} \sum_{k=0}^{n} \ell K_{M_{\delta}2k} \delta^{2k}$

$$\sum_{k=0}^{n} T_{2k}^{*} \delta^{2k} = \frac{\rho d^3}{m} \sum_{k=0}^{n} \ell (J_{L_{\delta}2k} - k_1^{-2} J_{T_{\delta}2k}) \delta^{2k}$$

and those terms appearing in the right hand summations which have powers of δ greater than 2n are neglected.*

It should be remembered that these equations are derived on the assumption of a flat trajectory. The complications introduced by gravity will be considered in a later report. In any event for a large class of problems this simple procedure should allow reasonably good prediction of the motion of missiles acted by non-linear forces.

7. SUMMARY

1. A convenient expansion of the aerodynamic force system for large yaw has been obtained and the necessary geometric relations for these large yaws derived.
2. Relations for parameters of a linear equation which is equivalent to the actual non-linear equation have been derived.
3. These relations have been tested by actual firing tests and excellent internal consistency has been observed. Where wind tunnel measurements were available, good agreement has been obtained.

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* The effect of the terms in J_L which are higher order than δ^2 on the $J_L' \lambda$ term in Equation (20) has been neglected.